# Sections 14.3 – 15.2

Theorem 1: (Length of a Path)
Assume that $r(t)$ is differentiable and that $r'(t)$ is continuous on $[a,b]$. Then the length $s$ of the path $r(t)$ for $a \le t \le b$ is equal to

$$\boxed{s = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt}$$

Speed is the rate of change of distance traveled with respect to time $t$. The arc length function is

$$\boxed{s(t) = \int_a^t \|r'(u)\|\, du}$$

Speed at time $t$ is: $\boxed{\dfrac{ds}{dt} = \|r'(t)\|}$

## Section 14.3 Additional exercises

1. Find the speed at the time $t=4$ of $r(t) = \langle 2t+3, 4t-3, 5-t \rangle$.

$r'(t) = \langle 2, 4, -1 \rangle \Rightarrow \|r'(4)\| = \sqrt{2^2+4^2+(-1)^2} = \sqrt{4+16+1}$
$= \boxed{\sqrt{21}}$

2. Compute the length of the curve
$r(t) = \langle 2t, \ln t, t^2 \rangle$ over the interval $1 \leq t \leq 4$.

$$\text{length on } [1,4] = \int_1^4 \sqrt{2^2 + \left(\frac{1}{t}\right)^2 + (2t)^2}\, dt$$

$$= \int_1^4 \sqrt{4 + \frac{1}{t^2} + 4t^2}\, dt = \int_1^4 \frac{\sqrt{4t^2 + 1 + 4t^4}}{t}\, dt$$

$$= \int_1^4 \frac{\sqrt{(2t^2+1)^2}}{t}\, dt = \int_1^4 \frac{2t^2+1}{t}\, dt = \int_1^4 2t + \frac{1}{t}\, dt$$

$$= \frac{\cancel{2}t^2}{\cancel{2}} + \ln|t| \Big|_1^4 = (4)^2 + \ln(4) - 1 - \ln(1)$$

$$= 16 + 2\ln 2 - 1 - 0$$

$$= \boxed{15 + 2\ln 2}$$

3. Find an arc length parametrization of the cycloid with parametrization $r(t) = \langle t - \sin t, 1 - \cos t \rangle$.
$0 \le t \le 2\pi$

<u>Step 1</u>: Form the arc length

$$s = g(t) = \int_0^t \sqrt{(1-\cos u)^2 + (\sin u)^2}\, du$$

$$= \int_0^t (2 - 2\cos u)^{1/2}\, du$$

$$= \sqrt{2} \int_0^t (1 - \cos u)^{1/2}\, du$$

Let $u = 2x$
$dt = 2dx$

$$= \sqrt{2} \cdot 2 \int_0^{t/2} \left( \frac{1 - \cos(2u)2}{2} \right)^{1/2} dx$$

use that $\sin^2 x = \frac{1 - \cos 2x}{2}$

$$= 2\sqrt{2} \cdot \sqrt{2} \int_0^{2t} \sin u \, dx$$

$$= 4 \left[ -\cos u \right|_0^{t/2} = 4(-\cos t/2 + 1)$$

$$= 4(1 - \cos t/2).$$

Step 2: Because $\|r'(t)\| \geq 0$

$g(t)$ is an increasing function on $[0, 2\pi]$ and $g(t)$ is one to one on $[0, 2\pi]$. So, solving for $t$ gives:

$$s = 4\left(1 - \cos \tfrac{t}{2}\right) \Rightarrow \frac{s}{4} = 1 - \cos t/2 \Rightarrow$$

$$\cos t/2 = 1 - s/4 \Rightarrow \frac{t}{2} = \cos^{-1}\left(1 - \frac{s}{4}\right)$$

$$\Rightarrow t = 2\cos^{-1}(1 - s/4).$$

So, $g^{-1}(s) = 2\cos^{-1}(1 - s/4)$.

Step 3: Take the new parametrization

$$r_1(s) = r(g^{-1}(s)).$$

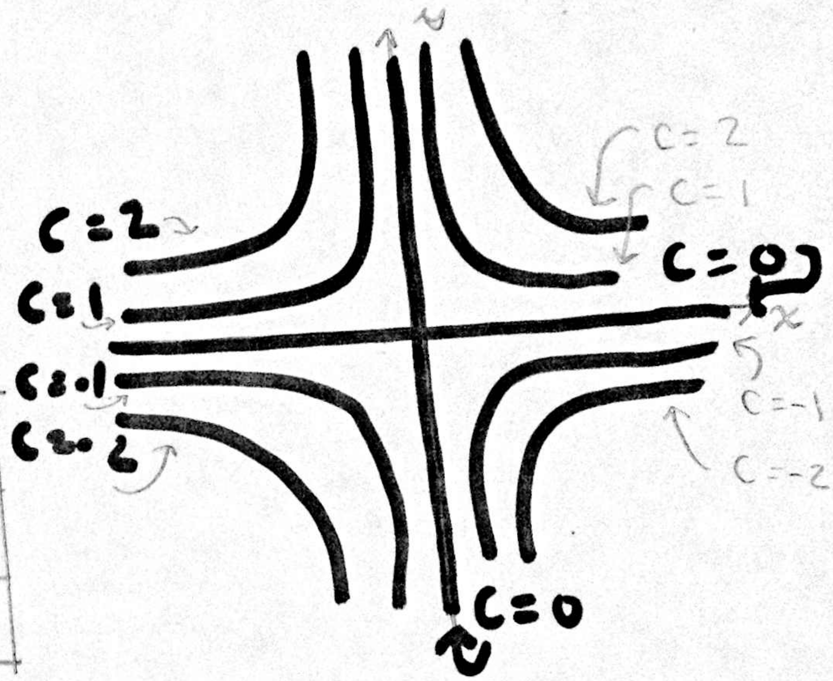
$$\boxed{r_1(s) = \left\langle 2\cos^{-1}(1 - s/4) - \sin\left(2\cos^{-1}(1 - s/4)\right), 1 - \cos\left(2\cos^{-1}(1 - s/4)\right) \right\rangle}$$

# Section 15.1 Additional Exercises

1. Draw a contour map of the following functions. Include at least five level curves:

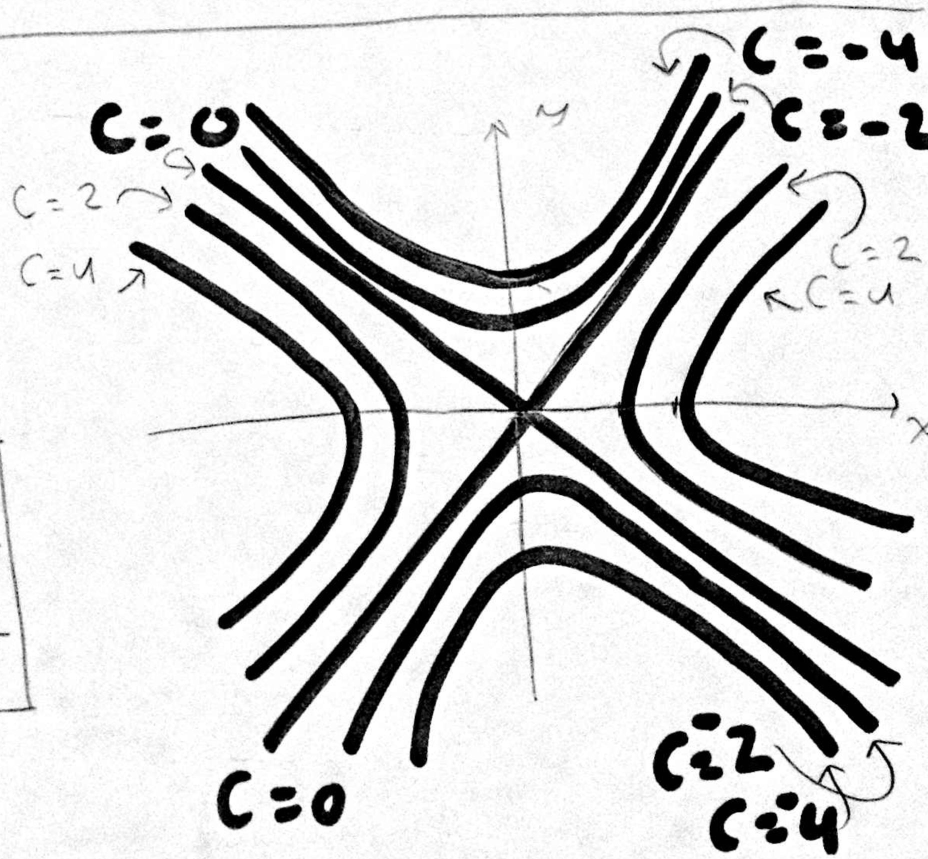
(i) $f(x,y) = xy$

| $f(x,y)=C$ | Plug in C, solve for y |
|---|---|
| $C=-2$ | $y=-2/x$ |
| $C=-1$ | $y=-1/x$ |
| $C=0$ | $y=0$ OR $x=0$ |
| $C=1$ | $y=1/x$ |
| $C=2$ | $y=2/x$ |




(ii) $f(x,y) = 3x^2 - y^2$

| $f(x,y)=C$ | Plug in C, solve for y |
|---|---|
| $C=-4$ | $y=\sqrt{3x^2+4}$ or $y=-\sqrt{3x^2+4}$ |
| $C=-2$ | $y=\sqrt{3x^2+2}$ or $y=-\sqrt{3x^2+2}$ |
| $C=0$ | $y=\pm\sqrt{3}\,x$ |
| $C=2$ | $y=\sqrt{3x^2-2}$, $y=-\sqrt{3x^2-2}$ |
| $C=4$ | $y=\sqrt{3x^2-4}$, $y=-\sqrt{3x^2-4}$ |



# Section 15.2 Additional Exercises



1. $$\lim_{(x,y)\to(2,-1)} (xy - 3x^2y^3) = (2)(-1) - 3(2)^2(-1)^3$$
$$= -2 - [3\cdot 4(-1)]$$
$$= -2 + 12$$
$$= \boxed{10}$$

We can plug in because $f(x,y) = xy - 3x^2y^3$ is continuous at the point $(2,-1)$.

2. $$\lim_{(x,y)\to(\pi/4,0)} \tan x \cos y \cancel{=} \left(\lim_{x\to\pi/4} \tan x\right)\left(\lim_{y\to 0} \cos y\right)$$
$$= \tan(\pi/4)\cos(0)$$
$$= \boxed{1}$$

We can plug in because $f(x,y)$ is a product of a function continuous in $x$ and a function continuous in $y$.

3. $$\lim_{(x,y)\to(0,0)} \frac{x}{x^2+y^2} = \text{does not exist.}$$

<u>Proof</u>: Along the $x$-axis we have:

$$\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{x}{x^2+0^2} = \lim_{x\to 0} \frac{1}{x}.$$ But

$$\lim_{x\to 0^+} \frac{1}{x} = \infty$$ and $$\lim_{x\to 0^-} \frac{1}{x} = -\infty.$$ So, $\hookrightarrow$

the limit along the x-axis does not exist and hence the overall limit does not exist.

4. $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{\sqrt{x^2+y^2}}$.

We can convert to polar coordinates and use the squeeze theorem.

$$\frac{x^2-y^2}{\sqrt{x^2+y^2}} = \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r} = r(\cos^2\theta - \sin^2\theta).$$

Then, $$0 \le \lim_{(x,y)\to(0,0)} \left|\frac{x^2-y^2}{\sqrt{x^2+y^2}}\right| = \lim_{r\to 0} r\,|\cos^2\theta - \sin^2\theta|$$

$$\le \lim_{r\to 0} 2r = 0.$$

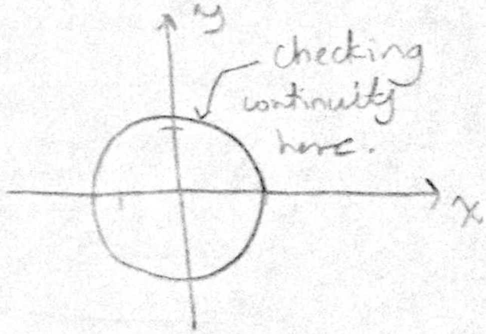
So, by the squeeze theorem,

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{\sqrt{x^2+y^2}} = 0}$$

5. Is the following function continuous?

$$f(x,y) = \begin{cases} x^2+y^2 & \text{if } x^2+y^2<1 \\ 1 & \text{if } x^2+y^2 \geq 1 \end{cases}$$

$f(x,y)$ is defined by a polynomial in the domain $x^2+y^2<1$, hence $f$ is continuous in this domain. In $x^2+y^2>1$ $f$ is the constant function 1 hence it is continuous in this domain also. So, we must check continuity at the points on the circle $x^2+y^2=1$




Expressing $f(x,y)$ in polar coordinates:

$$f(r,\theta) = \begin{cases} r^2 & \text{if } r<1 \\ 1 & \text{if } r>1 \end{cases}.$$

Since $\lim\limits_{r\to 1^-} f(r,\theta) = \lim\limits_{r\to 1^-} r^2 = 1$ and

$\lim\limits_{r\to 1^+} f(r,\theta) = \lim\limits_{r\to 1^+} r^2 = 1 \Rightarrow$ $f(r,\theta)$ is continuous at $r=1 \Rightarrow f(x,y)$ is continuous on circle $x^2+y^2=1 \Rightarrow f(x,y)$ is continuous.

6. Evaluate the limit $\lim_{(x,y)\to(0,2)} (1+x)^{y/x}$.

Let $L = \lim_{(x,y)\to(0,2)} (1+x)^{y/x}$. Then,

$$\ln(L) = \lim_{(x,y)\to(0,2)} \frac{y}{x}\cdot \ln(1+x) = \left(\lim_{(x,y)\to(0,2)} y\right)\left(\lim_{(x,y)\to(0,2)} \frac{\ln(1+x)}{x}\right)$$

$$= \lim_{y\to 2} y \cdot \lim_{x\to 0} \frac{\ln(1+x)}{x} = 2 \lim_{x\to 0} \frac{\ln(1+x)}{x}$$

L.H. rule "$\frac{0}{0}$"

$$= 2 \lim_{x\to 0} \frac{1}{1+x}$$

$$= 2.$$

$$\Rightarrow \boxed{L = e^2}$$